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Subsidence and Compaction Volumes

Antony Mossop

Abstract

For the case of a contracting body in an homogeneous, isotropic, linear elastic half-space the relationship between the 'subsidence volume' (i.e. the displacement volume of the free surface) and the internal in-situ volume change is not straightforward. It has also been incorrectly stated in the literature. In this paper the in-situ volume strain is analyzed and an expression for the in-situ volume strain Green's function is derived. It is found that the in-situ volume strain can be significantly larger than has been reported - for the case of a shallow, laterally extensive body the factor can be as great as $2(1-\nu)$.

1 Introduction: Surface Displacements and Subsurface Strains

In the geosciences, surface displacements of the Earth are frequently used to determine, or at least constrain, subsurface deformation. Examples include investigations of: tectonic fault slip, volcanic magma emplacement, fluid extraction and injection operations, mining operations, hydraulic fracturing, *etc.* Equally, the equivalent forward problem, that is the prediction of ground movements due to subsurface strains is also a matter of practical importance, for the purposes of planning and assessing potential environmental impacts on civil infrastructure and natural habitats.

Of particular interest to the oil and gas industry and the regulatory authorities charged with their oversight, is the surface deformation, particularly ground subsidence, associated with fluid (oil/gas) production from subsurface reservoir formations. As pore fluid pressures are reduced in the reservoir rock, it responds by contracting, and the resulting volume strain creates a pattern of downward and inward surface displacement. This paper deals with some of the elastostatic theory that underpins how this process is generally modeled in simulations.

Most of the present methods used to model and predict surface subsidence due to fluid extraction, are essentially based on the same methodology that Geertsma outlined in his seminal papers on the subject [1973a; 1973b]. In these, the subsurface is represented by a homogeneous, isotropic, linear elastic [HILE] halfspace and the displacements are calculated using a Green's function for a 'centre of compression nucleus of strain' in such a domain, appropriately scaled and integrated over the reservoir volume. The solutions though, yield what might be perceived as a counterintuitive result: upon analysis the apparent subsidence displacement volume is seemingly larger than the subsurface contraction volume. The origin of this article comes from requests I received, from the steering committee, to explain this phenomenon.

It should be noted at the outset that problems of this class crop up relatively frequently, the generic response is that a lot can be hidden by negligible perturbations at an infinite boundary condition, and the matter dismissed without much further thought (any finite number divided by infinity is, after all, zero). Or to rephrase that more formally, that there are an infinite number of functions that can have a finite volume integral while satisfying the condition of being zero at an infinite domain boundary.

On investigating the matter further though, a somewhat more complex story unfolds which highlighted some subtle misconceptions about volume strain; in particular that the relationship between the stress free strain and the constrained strain depends on the domain under consideration - although the finite impact of infinitesimal perturbations at an infinite boundary still plays its part. It should be noted that a careful analysis of the work by Seo and Mura [1979], Nowacki [1986] and Rudnicki [2002], all of which analyse the problem of volume strain within an inclusion in a HILE half-space domain, lead to very similar conclusions. This report though, investiagtes the matter explicitly.

2 Isotropic Volume Strain in a HILE Full-Space

The fundamental solution for this study (and one of the most important solutions in elastostatics) is that for a force acting at a point in a HILE full-space [Thomson, 1848]. The HILE properties of the material allow composite forces, acting at a point or points, to be constructed by simple operations of scaling, rotating and superposing this solution. A number of practically useful point force combinations can be generated in this manner, e.g. the force double or dipole, which is two forces of equal magnitude, but acting in opposite directions, in the limit that the distance between the two action points approaches zero. The general name for such point strain sources is ‘nuclei of strain’ [Love, 1927].

The displacement field that results from such nuclei of strain body forces, can be found by solving the elastostatic equations (alternatively called the Navier, Navier-Cauchy or Navier-Lamé equations)

$$G\nabla^2\mathbf{u} + \frac{G}{1-2\nu}\nabla\nabla\cdot\mathbf{u} = -\mathbf{b} \tag{1}$$

subject to the appropriate boundary conditions; here G is the shear modulus, $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the displacement vector field as a function of space $\mathbf{x}$, ν is Poisson's ratio and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ is the body force vector field. In this instance, where the domain is a full-space, the relevant boundary conditions are that displacements are zero at the infinite limits of the domain, i.e. $\mathbf{u}(\mathbf{x}) \to 0$ for $\|\mathbf{x}\| \to \infty$.

The strain, $\mathbf{e}$, and stress, $\boldsymbol{\sigma}$, tensor fields in a HILE medium are defined as follows

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{2}$$

$$\boldsymbol{\sigma} = 2G\mathbf{e} + \left(K - \frac{2G}{3}\right)\mathrm{tr}(\mathbf{e})\mathbf{I} = 2\mathrm{G}\left(\mathbf{e} + \frac{\nu}{1-2\nu}\mathrm{tr}(\mathbf{e})\mathbf{I}\right) \tag{3}$$

where the superscript ^{T} denotes the transpose operator, K is the bulk modulus, tr is the trace operator, and $\mathbf{I}$ is the unit tensor. Note that the stress is the deviatoric strain scaled by the shear modulus, G, plus an isotropic component equivalent to the volume strain ($\mathrm{tr}(\mathbf{e})$) scaled by the bulk modulus, K (it is often convenient to consider deviatoric and volumetric processes separately).

Of particular interest for this study is the point source isotropic volume strain, referred to as a ‘centre of compression/dilatation nucleus of strain’, it is composed of three orthogonal force doubles, of equal magnitude and polarity, acting on the same point. In short, it is an isotropic point expansion or contraction set of forces, that can be used as a Green function. As mentioned in the introduction, a distribution of such point source volume strain Green's functions, appropriately scaled and spatially superposed, can be used to approximate contracting inclusions in many solid-mechanics/geomechanics models.

To determine the magnitude of the body forces for such a centre of compression/dilatation, initially consider a stress free isotropic volume strain, $\bar{\mathbf{e}}$, for a small inclusion of material in the subdomain $\bar{\mathbf{x}}$, (i.e. a volume change, unconstrained by its surrounding material, but with no corresponding change in stress), of magnitude $\bar{\varepsilon}$, hence $\bar{\mathbf{e}}(\mathbf{x} \in \bar{\mathbf{x}}) = (\bar{\varepsilon}/3)\mathbf{I}$; $\bar{\mathbf{e}}(\mathbf{x} \notin \bar{\mathbf{x}}) = \mathbf{0}$. Note that this will yield a discontinuous, and hence unphysical state, as the inclusion will no longer be the same size as the space it previously occupied. Therefore to satisfy continuity conditions a set of corresponding isotropic body forces need to be applied to constrain the inclusion to its original size,

$$\mathbf{b} = -K\nabla\bar{\epsilon} \tag{4}$$

where $\bar{\epsilon} = \bar{\epsilon}(\mathbf{x}) = \mathrm{tr}(\bar{\mathbf{e}}(\mathbf{x}))$. This follows directly from the equilibrium equations

$$\nabla \cdot \boldsymbol{\sigma} = -\mathbf{b} \tag{5}$$

and the definition of bulk modulus. This then defines the magnitude of the isotropic expansion/contraction body forces; and hence the resulting displacement vector field at elastostatic equilibrium must satisfy

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla\nabla \cdot \mathbf{u} = \frac{2(1+\nu)}{3(1-2\nu)}\nabla\bar{\epsilon} \tag{6}$$

and the boundary condition $\mathbf{u}(\|\mathbf{x}\| = \infty) = \mathbf{0}$.

An isotropic expansion/contraction set of body forces in a HILE full-space must yield an isotropic displacement field, which is therefore irrotational. In which case applying the fundamental theorem of vector calculus means that the displacement field can be expressed as solely the gradient of a scalar potential function, $\Phi = \Phi(\mathbf{x})$, (as solenoidal vector field terms will be zero).

$$\mathbf{u} = \nabla\Phi \tag{7}$$

This then leads to the following relationships

$$\nabla^2\nabla\Phi + \frac{1}{1-2\nu}\nabla\nabla\cdot\nabla\Phi = \frac{2-2\nu}{1-2\nu}\nabla(\nabla^2\Phi) = \frac{2(1+\nu)}{3(1-2\nu)}\nabla\bar{\epsilon} \tag{8}$$

and hence

$$\nabla^2\Phi = \nabla\cdot\mathbf{u} = \frac{1+\nu}{3(1-\nu)}\bar{\epsilon} \tag{9}$$

The solutions can also be given in succinct form by making use of the Galerkin vector stress function, $\mathbf{F} = \mathbf{F}(\mathbf{x})$, where the displacement field is defined by

$$2AG\mathbf{u} = 2(1-\nu)\nabla^2\mathbf{F} - \nabla\nabla\cdot\mathbf{F} \tag{10}$$

here A is a scaling factor that varies between authors in the published literature, in this paper the convention $A = 1$ will be used. Mindlin [Mindlin, 1936] derived Galerkin vector stress Green's functions for a number of nuclei of strain. For the case of the centre of compression in a HILE full-space at point $\mathbf{y}$ that is being considered here

$$\mathbf{F}^* = \frac{G}{2\pi}\frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\ln(\|\mathbf{x}-\mathbf{y}\| + x_3 - y_3)\hat{\mathbf{x}}_3 \tag{11}$$

[N.B. the inherent nonuniqueness of the Galerkin stress function and spherical symmetry of the isotropic strain source means that the directional index 3 can be replaced with 1 or 2, or any arbitrary direction, all will yield the same displacements, stresses and strains.]. Here $\mathbf{F}^* = \mathbf{F}^*(\mathbf{x};\mathbf{y})$, where the superscript, *, denotes a Green's function for this field, i.e. $f^* = f^*(\mathbf{x};\mathbf{y})$ is the f field over domain $\mathbf{x}$ due to some given infinitesimal point source acting at location $\mathbf{y}$

The resulting displacement field Green's function, $\mathbf{u}^* = \mathbf{u}^*(\mathbf{x};\mathbf{y})$, for a centre of compression in a HILE full-space is

$$\mathbf{u}^* = \frac{1}{4\pi}\frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} \tag{12}$$

3 Stress Free Strain and Constrained Strain in a HILE Full-Space

When considering volume strain for a centre of compression or dilatation, one of the first concepts that needs to be understood is that the stress free isotropic strain at the nucleus location, $\bar{\mathbf{e}}(\mathbf{y})$, will not be the same as the strain that will actually occur in-situ, constrained by the surrounding material (except in the pathological case of an incompressible medium). In reality, the term 'nucleus of strain' is somewhat of a misnomer, as they are in fact defined as nuclei of forces. The stress free strain is simply the strain that would occur without the external constraint of the surrounding substrate, it simply provides a convenient way of deriving the magnitude of the point forces, but it does not equate to the constrained strain at the nucleus point when it is at a state of elastostatic equilibrium within the material. The relationship between constrained strain and stress free strain, at the nucleus point $\mathbf{y}$ for a HILE full-space can be found by combining equations (2) and (9)

$$\mathbf{e}(\mathbf{y}) = \frac{1+\nu}{3(1-\nu)}\bar{\mathbf{e}}(\mathbf{y}) \tag{13}$$

For the case of a single nucleus of strain in a HILE full-space, the system is spherically symmetric and hence as the stress free strain tensor for a centre of compression is isotropic, so too is the constrained strain, i.e. $\mathbf{e}(\mathbf{y}) = (\mathrm{tr}(\mathbf{e}(\mathbf{y}))/3)\mathbf{I}$; $\bar{\mathbf{e}}(\mathbf{y}) = (\mathrm{tr}(\bar{\mathbf{e}}(\mathbf{y}))/3)\mathbf{I}$. However, if another centre of compression is introduced, or a non-spherically symmetric distribution of such centres, then the symmetry is broken. In these circumstances the constrained strain will no longer be isotropic. Eshelby [1957] famously explored the problem of ellipsoidal inclusions undergoing isotropic stress free strain (i.e. a uniform distribution of centres of compression over an ellipsoidal volume) in a HILE full-space, where he derived analytic solutions for the constrained strain as a function of the ellipsoid aspect ratios. For more arbitrary distibutions of centres of compression though, the problem becomes intractable to solve analytically. However, while the deviatoric components of the constrained strain tensor might be sensitive to the distribution of centres of compression, the trace, or isotropic component of the constrained strain is actually completely independent of the distribution. Surprisingly the constrained volume strain at a point in a distribution of centres of compression, within a HILE full-space is always a factor of $(1+\nu)(3(1-\nu))$ times the stress free volume strain at that point. This fixed relationship is also apparent from equations (2) and (9) where for the general case

$$\mathrm{tr}(\mathbf{e}) = \frac{1+\nu}{3(1-\nu)}\bar{\epsilon} \tag{14}$$

A clearer understanding of why this is so can perhaps be gained by considering the Galerkin vector stress function for a centre of compression. Combining (2) and (10) gives an expression for the trace of the strain tensor

$$\mathrm{tr}(\mathbf{e}) = \frac{1-2\nu}{2G}\nabla \cdot \nabla^2 \mathbf{F} \tag{15}$$

Inserting the Galerkin stress Green's function for a centre of compression in a HILE full-space, (11), yields

$$\mathrm{tr}(\mathbf{e}(\mathbf{x} \neq \mathbf{y})) = 0 \tag{16}$$

i.e. a centre of compression in a HILE full-space exerts no volume strain outside of itself - although it can exert a deviatoric strain.

4 Volume Strain in a HILE Half-Space

For the purposes of modeling ground surface deformation due to subsurface poroelastic reservoir contraction, consideration must now turn to the problem of displacements due to volume strain within a half-space. As mentioned in the introduction, the standard methodology, originally outlined by Geertsma [Geertsma, 1973] which in turn applies principles developed to solve equivalent problems in thermoelasticity [Goodier, 1937; Nowacki, 1962], is to linearize the problem and use a point source volume strain Green's function, spatially superposed and scaled to approximate the contracting reservoir inclusion. The fundamental Green's function used is the centre of compression 'nucleus of strain' for the HILE halfspace, the solution of which was described in two independent papers by Mindlin & Cheng, and Sen, published almost simultaneously [Mindlin & Cheng, 1950; Sen, 1951].

The general approach for solving elastostatic problems of displacements due to body force distributions in a HILE half-space is to derive the solution for a HILE full space and then apply auxilliary conditions to satisfy the free surface boundary condition of zero traction. The uniqueness theorem of elastostatic equilibrium [Kirchoff, 1859] guarantees that such a solution is unique for the domain of interest. The auxilliary conditions required to satisfy the zero stress free surface boundary condition for a centre of compression nucleus of strain in a HILE half-space can be considered to be equivalent to that of a full space domain with appropriate 'mirror' force nuclei placed at the reflection point of the centre of compression in the 'free surface plane' (e.g. for a free surface plane at $x_3 = 0$, in a cartesian coordinate system, and a centre of compression at point $\mathbf{y} = (y_1, y_2, y_3)$, the auxilliary condition, $\boldsymbol{\sigma}(x_3 = 0) = \mathbf{0}$, is met by placing appropriate force nuclei at $\mathbf{y}' = (y_1, y_2, -y_3)$). The mirror nuclei of strain for a source centre of compression of unit magnitude can be decomposed into: i) a force double, acting in the direction normal to the free surface (in this case, $\hat{\mathbf{x}}_3$), scaled by a factor of 2; ii) a center of compression scaled by $(1 - 4\nu)$; iii) a doublet with axis normal to the free surface (in this case, $\hat{\mathbf{x}}_3$), scaled by the distance between the source and mirror points (in this case, $-2y_3$) [Mindlin and Cheng, 1950]. Note that a doublet is a source of compression and a source of dilatation, of equal magnitude, in the limit where the distance between them approaches zero.

The Galerkin vector stress Green's function for the combined source and mirror nuclei is given by

$$\mathbf{F}^* = \frac{G}{2\pi}\frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\Bigg[\ln(\|\mathbf{x}-\mathbf{y}\| + x_3 - y_3) + (1-4\nu)\ln(\|\mathbf{x}-\mathbf{y}'\| + x_3 + y_3) + \frac{2(x_3+y_3)}{\|\mathbf{x}-\mathbf{y}'\|} - \frac{2y_3}{\|\mathbf{x}-\mathbf{y}'\|}\Bigg]\hat{\mathbf{x}}_3 \quad (17)$$

Inserting this into (10) gives a displacement vector field Green's function of

$$\mathbf{u}^* = \frac{1}{4\pi}\frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\left[\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} + (3-4\nu)\frac{\mathbf{x}-\mathbf{y}'}{\|\mathbf{x}-\mathbf{y}'\|^3} - \right.$$

$$\left. 6x_3(x_3+y_3)\frac{\mathbf{x}-\mathbf{y}'}{\|\mathbf{x}-\mathbf{y}'\|^5} - (4x_3+6y_3-8\nu(x_3+y_3))\frac{\hat{\mathbf{x}}_3}{\|\mathbf{x}-\mathbf{y}'\|^3}\right] \quad (18)$$

The corresponding displacement field on the surface $x_3 = 0$ is hence

$$\mathbf{u}^* \,|_{x_3=0} = \frac{1+\nu}{3\pi}\bar{\varepsilon}\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\bigg|_{x_3=0} \quad (19)$$

For this case then, of a centre of compression in a HILE half-space, the Green's function for the displacement volume of the free surface (i.e. the 'subsidence volume' Green's function) is therefore given by

$$\frac{(1+\nu)}{3\pi}\bar{\varepsilon}\int_0^\infty\int_0^\infty \frac{y_3}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{2(1+\nu)}{3}\bar{\varepsilon} \quad (20)$$

and the idenity (14) *implies* (note emphasis, as it is erroneous)

$$\frac{(1+\nu)}{3\pi}\bar{\varepsilon}\int_0^\infty\int_0^\infty \frac{y_3}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 2(1-\nu)\mathrm{tr}(\mathbf{e}(\mathbf{y})) \quad (21)$$

This apparently anomalous result, that the surface displacement volume is larger than the in-situ volume strain of the centre of compression (except for the pathological case of an incompressible fluid, $\nu = 0.5$), was the original motivation for this study and was noted by Geertsma [Geertsma, 1973b]. However, it should be realised that while the displacement volume of the surface at $x_3 = 0$ may seem intuitively to be equivalent to the volume change within the half-space domain, that human intuition can often be misleading.

As a thought experiment, consider two, equal, centres of compression in a HILE full-space located at $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{y}' = (y_1, y_2, -y_3)$. For this example the displacement normal to the surface at $x_3 = 0$, is zero, and hence the surface displacement volume is also zero, yet there is clearly a net change in volume for any domain that contains one, or both, of the centres of compression. As has already been shown, a centre of compression in a HILE full-space cannot exert a volume strain ouside of itself and so it follows that the volume strain in any domain that contains one centre of compression is constant, e.g. for the set of domains

$$a_1 \leq x_1 \leq b_1; a_2 \leq x_2 \leq b_2; a_3 \leq x_3 \leq b_3 \tag{22}$$

where

$$-\infty \leq a_1 < y_1;\ y_1 < b_1 \leq \infty;\ -\infty \leq a_2 < y_2;\ y_2 < b_2 \leq \infty;\ 0 \leq a_3 < y_3;\ y_3 < b_3 \leq \infty \tag{23}$$

(i.e. the rectilinear domains within the half-space $x_3 \geq 0$ that contain the point $\mathbf{y}$)

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \mathrm{tr}(\mathbf{e}(\mathbf{x}))\ \mathrm{d}x_3\ \mathrm{d}x_2\ \mathrm{d}x_1 = \theta(\mathbf{y}) \tag{24}$$

where $\theta(\mathbf{y})$ is the net volume change associated with the point of compression at point y

$$\theta(\mathbf{y}) = \lim_{\Omega(\mathbf{y}) \to 0} \iiint_{\Omega(\mathbf{y})} \mathrm{tr}(\mathbf{e}(\mathbf{x}))\ \mathrm{d}x_3\ \mathrm{d}x_2\ \mathrm{d}x_1 \tag{25}$$

and $\Omega(\mathbf{y})$ denotes a volume about the point $\mathbf{y}$. Similarly for any convex domain containing the centre of compression point $\mathbf{y}$, with $x_3 \geq 0$ and a bounding surface on the plane $x_3 = 0$, the volume change is similarly $\theta(\mathbf{y})$. In this case there is zero displacement normal to the plane $x_3 = 0$, and the volume change must therefore be accommodated by displacement of the other bounding surfaces to the domain. Even as the domain expands to the limiting case of the half-space, $x_3 \geq 0$, the total volume change will remain constant, it is, as described in the introduction of this article, a case where inifinitesimal perturbations at an infinite boundary, when integrated equate to a finite value (c.f. Cantor's paradox).

To generate the solution for a centre of compression in a HILE half-space from this twin pair of centres of compression in a HILE full-space, we need to augment the centre of compression at $\mathbf{y}'$ with: another centre of compression scaled by -4ν; a doublet with axis normal to the free surface scaled by $-2y_3$; and a force double in the $\hat{\mathbf{x_3}}$ direction scaled by 2. The displacement at $x_3 = 0$ in the $\hat{\mathbf{x_3}}$ direction, given by (19) is therefore purely due to these augmentary nuclei of strain.

It has already been shown that the additional centre of compression at $\mathbf{y}'$ cannot induce any volume strain in the half-space $x_3 \geq 0$, and the doublet source at $\mathbf{y}'$, being a composite of negative and positive centres of compression, similarly cannot influence volume strain in the $x_3 \geq 0$ domain (more precisely, any combination of centres of compression at point $\mathbf{y}$ induces zero volume strain in the domain $\mathbf{x} \neq \mathbf{y}$). However, the force double at $\mathbf{y}'$ is a different matter, this has a deviatoric component (it can be decomposed into an isotropic and deviatoric set of forces), and while an isotropic set of forces cannot induce volume strain at a distance, a deviatoric set of forces can.

Using the identity for the Galerkin vector stress Green's function for the combined nuclei (17) and the volume strain expression (15), it follows that the volume strain Green's function within the half-space domain $x_3 > 0$ is

$$\mathrm{tr}(\mathbf{e}^*) = \frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\left[\delta(\mathbf{x}-\mathbf{y}) + \frac{1-2\nu}{\pi}\left(\frac{3(x_3+y_3)^2}{\|\mathbf{x}-\mathbf{y}'\|^5} - \frac{1}{\|\mathbf{x}-\mathbf{y}'\|^3}\right)\right] \tag{26}$$

where δ is the Dirac delta function. More explicitly, the volume strain Green's function at location $\mathbf{y}$ can be expressed as

$$\mathrm{tr}(\mathbf{e}^*(\mathbf{x}=\mathbf{y};\mathbf{y}) = \frac{1+\nu}{3(1-\nu)}\bar{\varepsilon}\left[\delta(\mathbf{x}-\mathbf{y}) + \frac{1-2\nu}{4\pi\|\mathbf{x}-\mathbf{y}'\|^3}\right] \tag{27}$$

This has significant and profound implications. Firstly it shows that the in-situ volume strain for a centre of compression in a HILE half-space is not the same as that for a HILE full-space. The assumption that the in-situ volume strain is described by (15), given by Geertsma in its equivalent form as $c_m\Delta p$ [Geertsma, 1973] (where c_m is the uniaxial compressibility, i.e. $c_m = 1/M$, i.e. the reciprocal of M the p-wave or uniaxial modulus, defined as $M = 3K(1-\nu)/(1+\nu)$, and Δp is the pore pressure change) is therefore erroneous. The free surface boundary condition affects the in-situ constrained strain, so that the constrained strain for a half space is not the same as the constrained strain for a full space. This is more clearly understood when consideration is given to the fact that the term 'nucleus of strain' is somewhat of a misnomer - the nuclei are in fact defined in terms of equivalent body forces. The resulting strains induced by these body forces are therefore dependent on the boundary conditions. For the case of a half space, the relationship between stress free volume strain and in-situ constrained volume strain will depend on distance from the free surface. Hence a uniform stress free strain in an ellipsoidal inclusion in a HILE half-space will not result in a uniform in-situ constrained strain within the inclusion domain, as might be assumed from Eshelby theory [Eshelby, 1957; Seo & Mura, 1979; Rudnicki, 2002]. In fact it follows that the in-situ strain for a single infinitesimal point centre of compression in a HILE half-space won't even be isotropic.

Secondly, it is follows that volume strain occurs in the half-space $x_3 \geq 0$ oustide of the centre of compression. By extension it follows that for a distribution of centres of compression, in A HILE half-space, representing a compacting reservoir, that volume strain will also occur outside of the reservoir. Hence determining net volume strain within the half-space domain requires integration of a non-trivial, spatially dependent function over the entire domain.

The effects that are discussed here, due to the difference between full-space and half-space boundary conditions, are relatively minor if the dimensions of the contracting reservoir volume are small compared to the distance from the free surface. As the reservoir depth-to-size ratio becomes large, the in-situ strains and displacements will approach those of a full space and the relationships given in (14) and (21) will be reasonably accurate. However, for laterally extensive, shallow reservoirs, where the depth-to-size ratio becomes small, the influence of the free surface will rapidly become very significant. In the limit of an extensive reservoir at the surface, with uniform stress free volume strain of magnitude, $\bar{\varepsilon}$ the in-situ volume strain approaches

$$\mathrm{tr}(\mathbf{e}^*(\bar{\mathbf{x}})) \to 2(1-\nu)\frac{1+\nu}{3(1-\nu)}\bar{\varepsilon} \tag{28}$$

a factor of $2(1-\nu)$ larger than is the case for a full-space (or a very deep reservoir).

5 Conclusions

The relationship between the 'subsidence volume' (i.e. the displacement volume of the free surface) and the internal in-situ volume change for a contracting inclusion within a HILE half-space is not as straightforward as it would at first appear nor is it as stated by Geertsma [Geertsma, 1973b]. The problem can be analyzed by considering the elementary solution for a single centre of compression nucleus of strain in a HILE half-space domain (the contracting inclusion can constructed from this elementary solution by superposition). The solution for a centre of compression in a HILE half-space can be represented by the following set of nuclei in a HILE full space: i) an identical centre of compression at the same co-ordinates in a HILE full-space; ii) a centre of compression of the same magnitude at the reflection point given by replacing the free surface with a plane of reflection symmetry; iii) additional centre of compression, doublet and force double nuclei at this reflection point. Analysis of the combination of i) and ii) (i.e. two identical centres of compression about a plane of reflection) makes it clear that volume strain can be accomodated by negligible displacements at the infinite boundary, that can still satisfy the boundary conditions, and yet cause zero displacement volume of the virtual free surface. Hence, displacement volume and internal volume change do not have to be equal. Further analysis though, shows an even more complex story, where the internal volume change is found to be a function of the size and location of the contracting inclusion, which can be determined by appropriate superposition of the volume strain Green's function given by identity (26).

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