Report

Erratum to the report:
"Research and Development project for Geodetic Deformation Monitoring"

Revision of recommendations from the project:
"Long-term study on anomalous time-dependent subsidence in the Wadden sea region"

Sami Samiei-Esfahany
Department of Geoscience and Remote Sensing
Delft University of Technology, Netherlands

Hermann Bähr
Nederlandse Aardolie Maatschappij B.V.

EP Document Nummer.:	***EP201701215912***		
Revision No.:	*1.0*		
Document Date:	***20-01-2017***	*Security:*	***Restricted***

Contents

1 Background **3**

2 Revising recommendation 1 from LTS 1 project **4**

2.1 Variogram modelling approach of LTS1 4

2.2 A mistake in LTS1 implementation 5

2.3 Results of the new processing 6

3 Revising recommendation 2 from LTS 1 project **10**

A Appendix: 2nd order statistics for double differences with fractional Brownian motion **12**

A.1 Fractional Brownian motion 12

A.2 2nd order statistics for double differences (DDs) 12

A.3 Variogram model for double differences 14

B References **17**

1 Background

In Samiei-Esfahany and Bähr (2015), we made six basic recommendations regarding the processing and exploitation of geodetic subsidence data for geomechanical modelling. Two of the recommendations require revision. These two recommendations read:

1. Use the proposed stochastic model of geodetic data in the form of full noise covariance matrices in geomechanical modelling

2. Use double differences with multiple reference epochs and points as an optimal interface between geodetic data and geomechanical modelling

In this document, we revise these two recommendations. Section 2 elaborates on a mistake in the methodology that had been applied for estimation of the stochastic-model parameters regarding the first recommendation. Section 2 focuses on the applicability of using double differences with multiple reference points/epochs as proposed in the second recommendation.

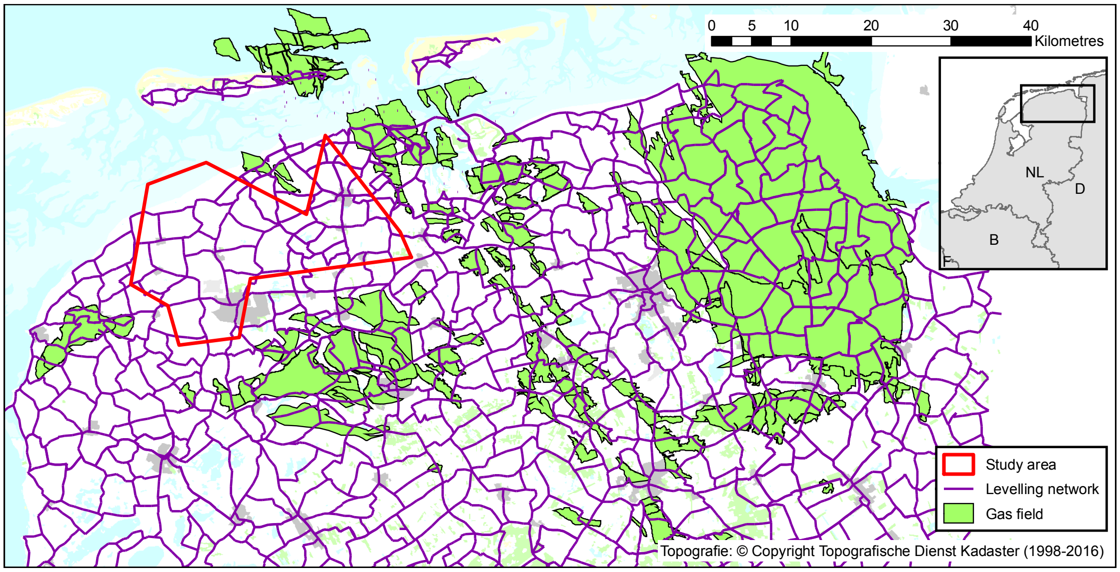

Figure 1: Selected area for levelling data analysis. For estimation of the idelaisation noise parameters, the levelling surveys outside the influence area of gas production have been selected.

2 Revising recommendation 1 from LTS 1 project

In Samiei-Esfahany and Bähr (2015), we proposed of using, in addition to the covariance matrix of measurement noise, the covariance matrix of the so-called idealisation noise. An stochastic model has been derived for the idealisation noise from levelling surveys outside the influence area of gas production where shallow deformation effects can be isolated from deep source subsidence (see Figure 1). The spatio-temporal stochastic analysis of the selected area showed the presence of both temporal non-stationary and spatially correlated idealisation noise components in the levelling dataset. We proposed an spatio-temporal variogram model for the idealisation noise, and we estimated the model parameters. However, the implementation of the estimation algorithm was not completely error-free. In the following, we clarify the problem, propose a corrected estimation procedure, and re-estimate the model parameters.

2.1 Variogram modelling approach of LTS1

The variogram model that had been used in Samiei-Esfahany and Bähr (2015) was based on random processes with fractional Brownian motions (See Yaglom (1962), and the Appendix A for more information on statistical properties of fractional Brownian motions). The parameters of the proposed model had been estimated by fitting the model to the empirical variograms computed from the levelling data in the selected/stable area. In levelling datasets, we are dealing with single difference (SD) height measurements. In order to exclude the height information and isolate the idealisation noise components, we computed the empirical variograms from double difference (DD) measurements as

$$\hat{\gamma}(\Delta t_{12}, h_{ij}) = \frac{1}{2}\Big((z_{ri}^{t_1t_2} - z_{rj}^{t_1t_2})^2\Big) = \frac{1}{2}\Big((z_{ij}^{t_1t_2})^2\Big), \tag{1}$$

where:
$z_{ri}^{t_1t_2}$ is the DD measurement as $z_{ri}^{t_1t_2} = (z_i^{t_2} - z_r^{t_2}) - (z_i^{t_1} - z_r^{t_1})$, and
$z_{rj}^{t_1t_2}$ is the DD measurement as $z_{rj}^{t_1t_2} = (z_j^{t_2} - z_r^{t_2}) - (z_j^{t_1} - z_r^{t_1})$,
$z_{ij}^{t_1t_2}$ is the DD measurement as $z_{ij}^{t_1t_2} = (z_j^{t_2} - z_i^{t_2}) - (z_j^{t_1} - z_i^{t_1})$,
$z_i^{t_1}$ is the idealisation noise components of point i at time t_1,

Δt_{12} is the time difference between two epochs as $\Delta t_{12} = |t_1 - t_2|$,
h_{ij} is spatial distance between points i and j, and
$\gamma(\Delta t, h)$ is the empirical variogram as function of Δt and h.

Assuming a fractional Brownian motion (Yaglom, 1962), the empirical variograms can be modeled as (see the proof in Appendix A)

$$\mathrm{E}\{\hat{\gamma}(\Delta t_{12}, h_{ij})\} = \frac{1}{2}\mathrm{E}\{(\underline{z}_{ij}^{t_1t_2})^2\} = \frac{1}{2}\mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} = \left(\sigma_s^2 - \sigma_s^2 e^{-\frac{h_{ij}}{L}}\right)\Delta t_{12}^{p_s} + \sigma_t^2 \Delta t_{12}^{p_t} \tag{2}$$

where:
$\mathrm{E}\{.\}$ and $\mathrm{D}\{.\}$ are expectation and dispersion operators,
$\underline{\hat{\gamma}}(\Delta t, h)$ is the empirical variogram as function of Δt and h,
σ_t^2 is the variance of the temporal component,
p_t is the power of the non-stationary signal associated with the temporal component,
σ_s^2 is the variance of spatio-temporal component,
L is the correlation length of the spatio-temporal component,
p_s is the power of the non-stationary signal associated with the spatio-temporal component.

Having the observations $\hat{\gamma}$ and the functional model of Eq.(2), the model parameters are estimated by nonlinear least squares estimation.

2.2 A mistake in LTS1 implementation

Note that the empirical variograms of Eq. (1) are, in fact, the spatial variograms per double difference epochs t_1t_2. However, in the modelling implementation of LTS1 project, the empirical cross-variograms had been also computed and included in the estimation. Cross-variograms have been computed as:

$$\hat{\gamma} = \frac{1}{2}\left((z_{ri}^{t_1t_2} - z_{rj}^{t_3t_4})^2\right), \tag{3}$$

where:
$z_{ri}^{t_1t_2}$ is the DD measurement as $z_{ri}^{t_1t_2} = (z_i^{t_2} - z_r^{t_2}) - (z_i^{t_1} - z_r^{t_1})$, and
$z_{rj}^{t_3t_4}$ is the DD measurement as $z_{rj}^{t_3t_4} = (z_j^{t_4} - z_r^{t_4}) - (z_j^{t_3} - z_r^{t_3})$.

The problem is that the model of Eq.(2) is not valid for cross-variograms. Therefore, including the cross-variograms in the modelling can result in wrong/biased estimates of the idealisation noise parameters.

The correct model for cross-variograms is (see the proof in Sec. A):

$$\begin{aligned}\mathrm{E}\{\frac{1}{2}(\underline{z}_{ri}^{t_1t_2} - \underline{z}_{rj}^{t_3t_4})^2\} &= \frac{1}{2}\mathrm{D}\{\underline{z}_{ri}^{t_1t_2} - \underline{z}_{rj}^{t_3t_4}\} \\ &= \sigma_s^2 \Delta t_{12}^{p_s}(1 - e^{\frac{-h_{ri}}{L}}) + \sigma_s^2 \Delta t_{34}^{p_s}(1 - e^{\frac{-h_{rj}}{L}}) - ... \\ &...\frac{1}{2}\sigma_s^2\left(\Delta t_{14}^{p_s} + \Delta t_{23}^{p_s} - \Delta t_{13}^{p_s} - \Delta t_{24}^{p_s}\right)\left(1 + e^{\frac{-h_{ij}}{L}} - e^{\frac{-h_{ri}}{L}} - e^{\frac{-h_{rj}}{L}}\right) + ... \\ &...\sigma_t^2 \Delta t_{t_1t_2}^{p_t} + \sigma_s^2 \Delta t_{t_3t_4}^{p_t} - \frac{1}{2}\sigma_t^2\left(\Delta t_{14}^{p_t} + \Delta t_{23}^{p_t} - \Delta t_{13}^{p_t} - \Delta t_{24}^{p_t}\right).\end{aligned} \tag{4}$$

Note that, when $t_3 = t_1$ and $t_4 = t_2$, the equation Eq.(4) reduces to the model of Eq.(2).

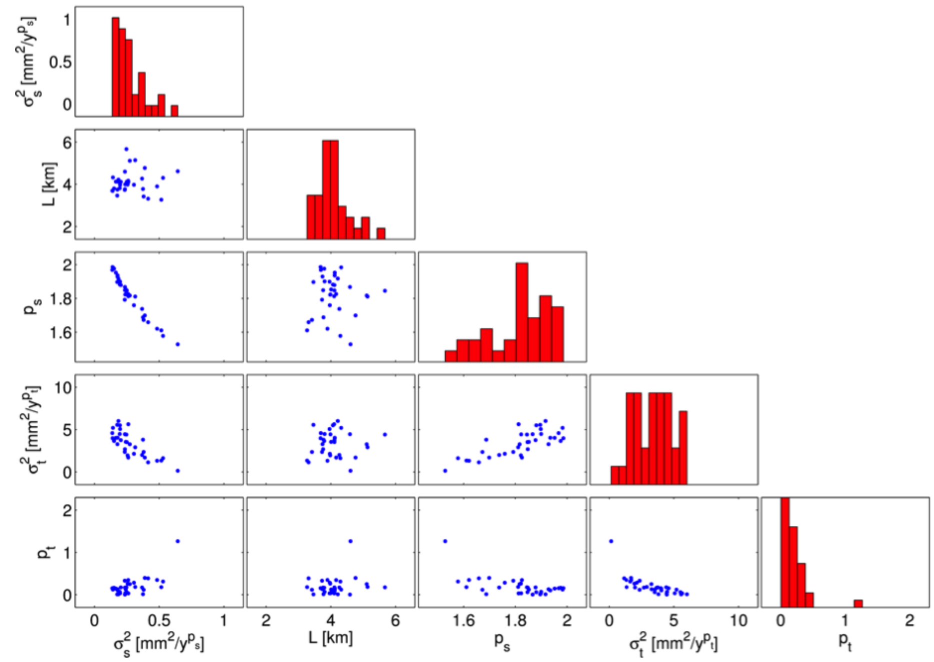

Figure 2: Results of the the Monte-Carlo approach to estimate the five model parameters. Histogram and mutual scatter plot of the estimates. Only 35 from 500 cases converged to a solution.

In order to have a correct estimate, either the cross-variograms should be excluded from the estimation, or the functional model should be extended to the model of Eq.(4). The second option requires a fully new implementation of the estimation process. Note that the model of Eq.(4) is not only a function of the two variables Δt_{12} and h_{ij} (as in the Eq.(2)), but it is a function of eleven variables (i.e., all the Δt and h variables).

In the next section, we present the new results based on the same functional model of Eq.(2)), but, this time, we do not include the emperical cross-variograms in the estimation.

2.3 Results of the new processing

For empirical variogram computation, the same approach as in Samiei-Esfahany and Bähr (2015) has been used. For estimation of the five parameters of the noise model of Eq. (2), we used nonlinear weighted least squares, where weights are assigned to empirical variograms based on the number of samples per bin. For solving the nonlinear least squares, the iterative trust-region-reflective algorithm (as implemented in Matlab2014A) was used (Coleman and Li, 1996). In order to obtain also some quality measure about the estimated parameters, we applied the estimation in a Monte-Carlo manner. That is, instead of estimating the parameters only once, we estimated them 500 times. Each time, we added a new realization of measurement noise to the original observations, followed by the computation of the experimental variograms and the non-linear parameter estimation. The mean of the 500 solutions gives the final estimates. Empirical dispersion or standard deviation of the 500 solutions provides a quality description. Note that, in nonlinear least squares, not all the 500 cases converged to a solution. Therefore, for computation of empirical standard deviations, only the converged solutions were used.

Applying the aforementioned methodology to estimate all the five parameters revealed that the information content of the empirical variograms is not enough to determine all the five model parameters uniquely. Figure 2 shows the histogram and mutual scatter plot of the Monte-Carlo sample estimates. In this analysis, only 35 from 500 cases converged to a solution, indicating the poor performance of the modelling. The results are summarized in Table 1. The empirical standard

Parameter [unit]	σ_s^2 [mm^2/km/y^{p_s}]	L [m]	p_s [-]	σ_t^2 [mm^2/y^{p_t}]	p_t [-]
Estimates	0.277	4086	1.81	3.36	0.194
Empirical std	0.123	513	0.121	1.497	0.219
Relative error	45%	13%	7%	45%	113%

Table 1: Estimated parameters for spatio-temporal idealisation noise model. The results of the first analysis to estimate all the five parameters.

deviation of some parameters is very large, resulting, for example, in 113 percent relative error for the parameter p_t. Also, the parameters σ_s^2 and σ_t^2 are not constrained very well and show relatively low precision. In order to get more precise and reliable estimates, we need to introduce some constraints in the model.

We chose to put a constraint on the p_t parameter, based on an external estimate of this parameter from another study (van Leijen et al., 2017). In that study, the parameters of temporally correlated idealisation noise have been estimated from levelling observations between very close benchmarks in Wadden sea region. Such close distances provide a unique opportunity to isolate the temporally correlated idealisation noise regarding the benchmarks temporal instabilities. In van Leijen et al. (2017), the parameter p_t has been estimated as 1.86. We fixed this parameter in the model and repeated the whole procedure again to estimate the remaining four parameters. The outcome is presented in Figure. 3 and Table 2. The results show much better performance that the first analysis. All the parameters show a normal-shape histogram, and the empirical standard deviations show better precision. Also, from 500 iterations, 465 times the algorithm converged to a solution.

The fitted model to empirical variograms using the estimates of Table 2 is plotted in Figure 4. To get a feeling about the spatio-temporal significance of the estimated idealisation noise model, we demonstrate standard deviations of the different noise model components for levelling double differences in Figure 5. Based on the estimated parameters, standard deviations of double difference measurement as a function of distance have been evaluated for time spans of 1, 5, 20, and 50 years. For comparison, the noise models are evaluated with both revised parameters and the old/wrong parameters of LTS1. Note that the measurement noise component here is just an arbitrary example. In practice, the contribution of levelling measurement noise is adaptive and depends on the levelling network configuration.

We conclude that the revised values (as indicated in Table 2) are the final estimated parameters of the proposed idealisation noise model.

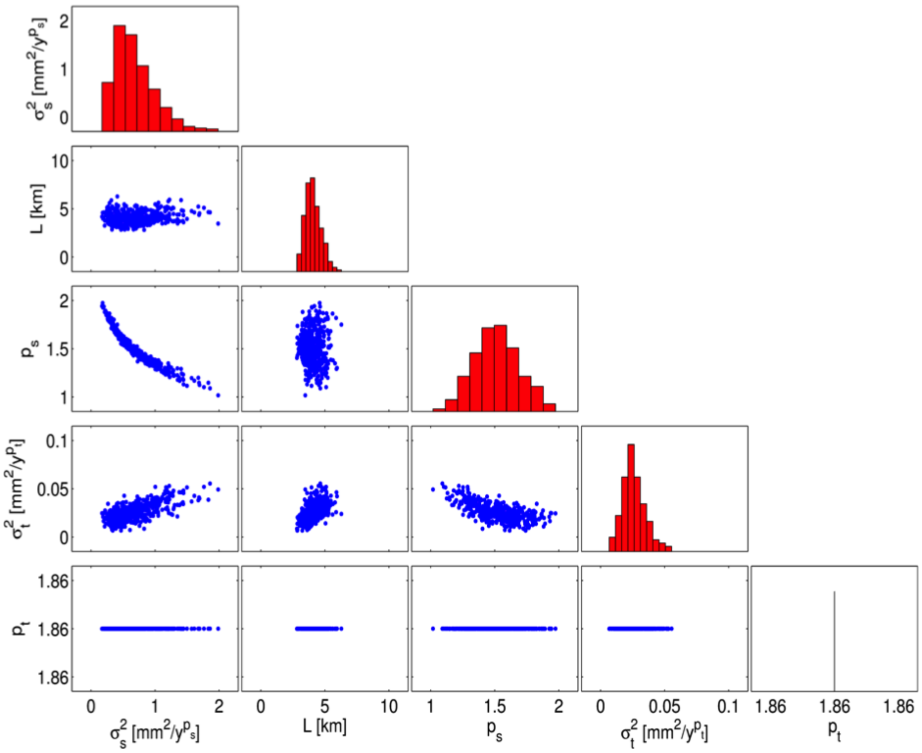

Figure 3: Results of the the Monte-Carlo approach to estimate only four model parameters. The parameter p_t has been constrained to be equal to 1.86. The plot shows the histograms and mutual scatter plot of the estimates. From 500 in total, 465 times the algorithm converged to a solution.

Parameter [unit]	σ_s^2 [mm^2/km/y^{p_s}]	L [m]	p_s [-]	σ_t^2 [mm^2/y^{p_t}]	p_t [-]
Estimates	0.684	4069	1.51	0.026	1.86
Empirical std	0.32	600	0.17	0.009	(constrained)
Relative error	47%	15%	12%	34%	-

Table 2: Estimated parameters for spatio-temporal idealisation noise model. The results of the second analysis to estimate only four parameters. The parameter p_t has been fixed 1.86 in the estimation.

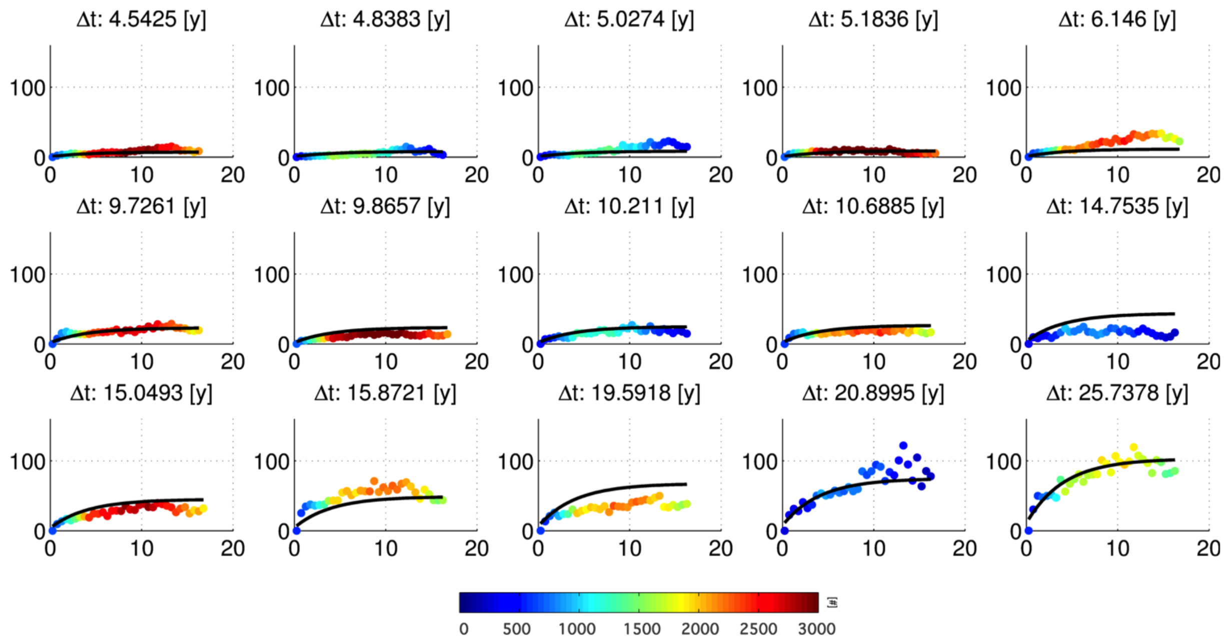

Figure 4: Spatial variogram profiles of levelling data over the signal-free area for different temporal lags (from 4.5 to 25.7 years). The X-axes are labelled by the spatial lag in [km], and the Y-axes display the variograms [mm2].

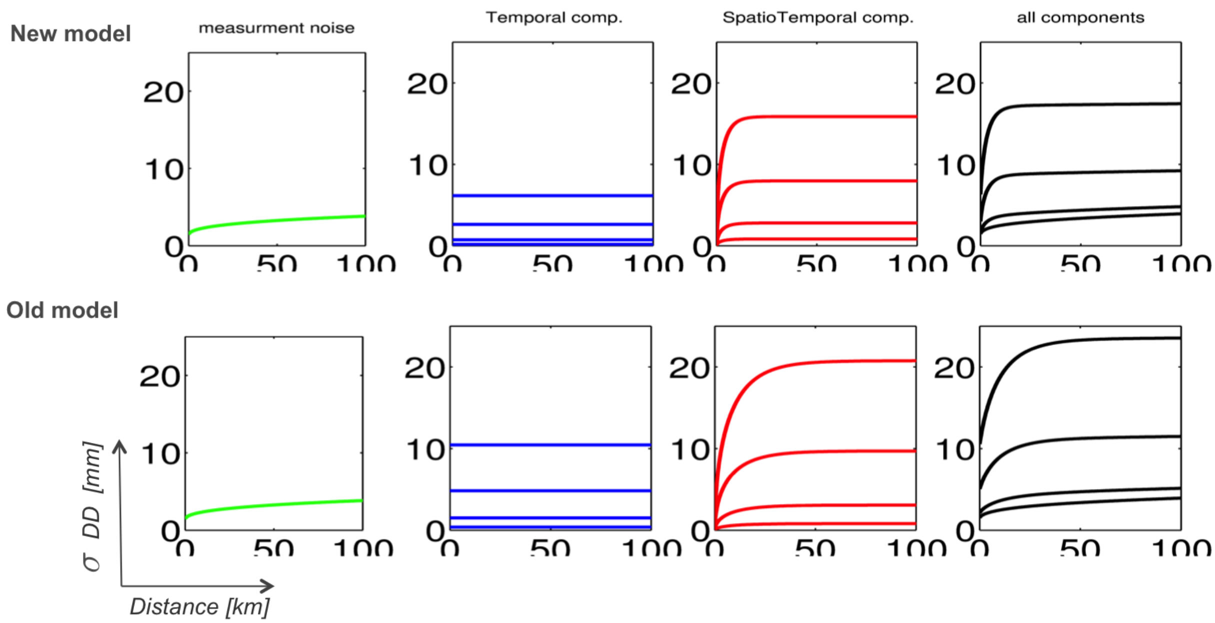

Figure 5: The standard deviation plot of different component of the proposed model evaluated for time spans of 1, 5, 20, and 50 years: Note that the measurement noise component here is just an arbitrary example for a levelling network. In practice the contribution of levelling measurement noise is invariant in time and adaptive to the levelling network configuration. For comparison, the noise models are evaluated with both revised parameters (top row) and the old and wrong parameters of LTS1 (bottom row). The new estimated parameters are listed in Table 2, and the old parameters read: $\sigma_s^2 = 0.651$ mm^2/km/y^{p_s}, $L = 12646$ m, p_s=1.66, $\sigma_t^2 = 0.148$ mm^2/y^{p_t}, $p_t = 1.68$.

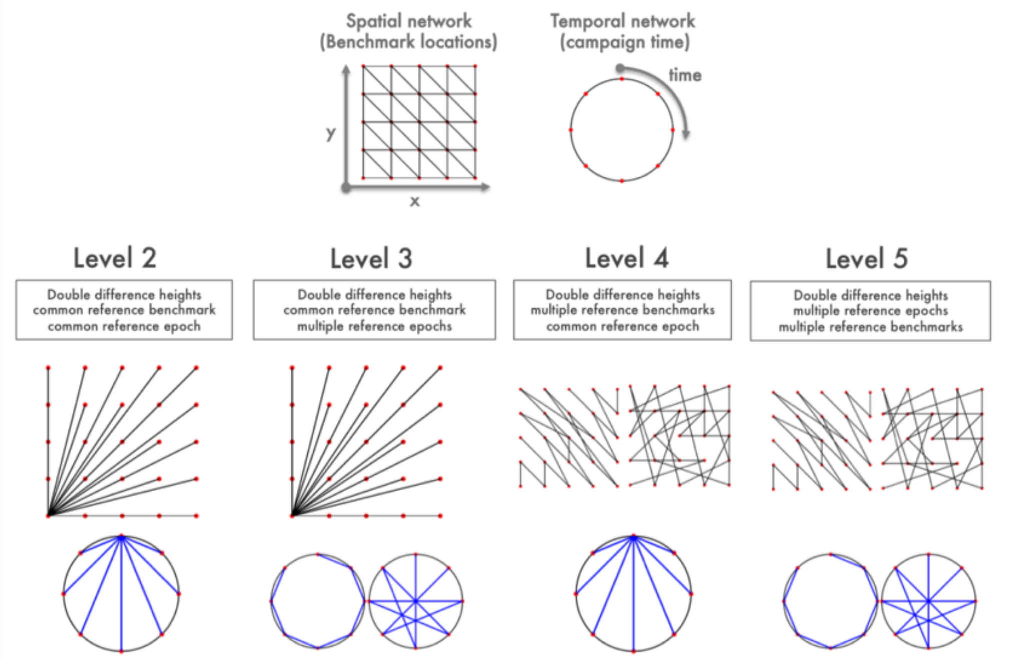

Figure 6: Schematic visualisation of output levels for levelling (as proposed in LTS1); The concept is adaptable for other techniques.

3 Revising recommendation 2 from LTS 1 project

In Samiei-Esfahany and Bähr (2015), the full recommendation regarding double differences:

> ”*Use double differences with multiple reference epochs and points as an optimal interface between geodetic data and geomechanical modelling*”.

During LTS1, the use of multiple reference points and epochs for the double differences in a dataset (i.e., output level 5,see Figure 6) was thought to be more optimal for two reasons:

1. Reduction of the sensitivity of estimates to covariances and thus reduction of the risk of biased estimates due to false assumptions in the stochastic model,
2. If a single reference point and a single reference epoch are used, the double-differences are not zero mean, which is assumed by the stochastic model.

Both reasons turned out to be invalid:

1. The reasoning was as follows: if covariances are neglected, then the level 5 (geomechanical) model estimate is closest to the simulated true value. If a single reference point and epoch are used, a common noise component is included in all double differences. This component is amplified by the multiple inclusions without considering stochastic dependencies. In the case of multiple points and epochs, this noise is mitigated, because it is different for every double difference. Neglecting covariances was thought to have a similar effect as using the full covariance matrix but with a biased covariance model. This is not true. Neglecting covariances in level 2 yields a stochastic model that is different from the model obtained by neglecting covariances in level 5. Transforming a (fully populated or diagonal)

covariance matrix between level 2 and level 5, however, does not alter the stochastic model between these two levels. Hence, the effect of a biased stochastic model cannot be mitigated by using level 5 instead of level 2.

2. If double differences are taken with respect to a single reference point and a single reference epoch, their mean is generally not zero. (This may happen by chance though.) If the double differences are used to characterize their stochastic properties in a variogram, this is a requirement. In that case, they are assumed to fully characterize the underlying stochastic process. If the double differences are used for a confrontation with geomechanical model predictions, however, they are only considered a realization of the stochastic process and not required to be zero mean. Hence, geomechanical model estimates are not sensitive to the used output level.

It can be concluded that the choice of the output level does not make a difference as long as the full covariance matrix is used.

A Appendix: 2nd order statistics for double differences with fractional Brownian motion

A.1 Fractional Brownian motion

For a random process (or noise component) with a fractional Brownian motion, the variance increases with time. The variance/dispersion of such a process (hereafter denoted by $\underline{z}$) at point i and time t_1 is (Yaglom, 1962):

$$\mathrm{D}\{\underline{z}_i^{t_1}\} = \sigma^2 t_1^p, \tag{5}$$

where $0 < p < 2$ is a power index. The p exponent describes the smoothness of the resultant motion, with a higher value leading to a smoother motion.

Based on the definition of Eq.(5) and using the error propagation law, the covariance between $\underline{z}_i$ at two different time t_1 and t_2 is computed as (Yaglom, 1962):

$$\mathrm{C}\{\underline{z}_i^{t_1}, \underline{z}_i^{t_2}\} = \frac{1}{2}\sigma^2(t_1^p + t_2^p - \Delta t_{12}^p), \tag{6}$$

where $\Delta t_{12} = |t1 - t2|$.

Extension to a spatially correlated signal
Assuming a random process with a fractional Brownian motion in the time domain, but also with correlation and 2nd order stationarity in the space domain, the model of Eq.(6) can be extended to the covariance between z at t_1 and t_2, and at two different locations i and j:

$$\mathrm{C}\{\underline{z}_i^{t_1}, \underline{z}_j^{t_2}\} = \frac{1}{2}\sigma^2(t_1^p + t_2^p - \Delta t_{12}^p)e^{-\frac{h_{ij}}{L}}, \tag{7}$$

where h_{ij} is the distance between the two points, and L is the spatial correlation length (Note that there are, in general, different models for explaining the spatially correlated signal. Here, as an example, we use the exponential covariance model $\sigma^2 e^{-h_{ij}/L}$).

Using Eqs. (6) and (7), the 2nd order statistics (i.e., variance and covariances) of the process z for different choices of i, j, t_1, and $t2$ can be evaluated. The Table 3 shows the overview of these 2nd order statistics for both spatially correlated and spatially uncorrelated processes.

A.2 2nd order statistics for double differences (DDs)

Using the linear error propagation law, the statistics of fractional Brownian processes (Table. 3), can be propagated to DD combinations.

For a general case, when we have two double differences $\underline{z}_{ij}^{t_1 t_2}$ and $\underline{z}_{kl}^{t_3 t_4}$, the functional relationship between double differences and un-differenced z components is written as:

Table 3: overview of these 2nd order statistics for both spatially correlated and uncorrelated fractional Brownian processes.

Statistics	Symbol	Temporally non-stationary Spatially correlated (2nd order statinonary)	Temporally non-stationary Spatially no correlation ($L \to \epsilon$)
$\mathrm{D}\{\underline{z}_i^{t_1}\}$	q_{ii}^{11}	$\sigma^2 t_1^p$	$\sigma^2 t_1^p$
$\mathrm{C}\{\underline{z}_i^{t_1}, \underline{z}_j^{t_1}\}$	q_{ij}^{11}	$\sigma^2 t_1^p e^{-\frac{h_{ij}}{L}}$	0
$\mathrm{C}\{\underline{z}_i^{t_1}, \underline{z}_i^{t_2}\}$	q_{ii}^{12}	$\frac{1}{2}\sigma^2(t_1^p + t_2^p - \Delta t_{12}^p)$	$\frac{1}{2}\sigma^2(t_1^p + t_2^p - \Delta t_{12}^p)$
$\mathrm{C}\{\underline{z}_i^{t_1}, \underline{z}_j^{t_2}\}$	q_{ij}^{12}	$\frac{1}{2}\sigma^2(t_1^p + t_2^p - \Delta t_{12}^p)e^{-\frac{h_{ij}}{L}}$	0

$$\begin{bmatrix} \underline{z}_{ij}^{t_1t_2} \\ \underline{z}_{kl}^{t_3t_4} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \underline{z}_i^{t_1} \\ \underline{z}_i^{t_2} \\ \underline{z}_j^{t_1} \\ \underline{z}_j^{t_2} \\ \underline{z}_k^{t_3} \\ \underline{z}_k^{t_4} \\ \underline{z}_l^{t_3} \\ \underline{z}_l^{t_4} \end{bmatrix}}_{Z}. \tag{8}$$

The dispersion of $\left[\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\right]^T$ is computed by linear error propagation as:

$$\mathrm{D}\{\begin{bmatrix} \underline{z}_{ij}^{t_1t_2} \\ \underline{z}_{kl}^{t_3t_4} \end{bmatrix}\} = \begin{bmatrix} \mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} & \mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} \\ \mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} & \mathrm{D}\{\underline{z}_{kl}^{t_3t_4}\} \end{bmatrix} = SQ_ZS^T, \tag{9}$$

where Q_Z is the covariance matrix of the vector Z:

$$Q_Z = \mathrm{D}\{Z\} = \begin{bmatrix} q_{ii}^{11} & q_{ii}^{12} & q_{ij}^{11} & q_{ij}^{12} & q_{ik}^{13} & q_{ik}^{14} & q_{il}^{13} & q_{il}^{14} \\ & q_{ii}^{22} & q_{ij}^{12} & q_{ij}^{22} & q_{ik}^{23} & q_{ik}^{24} & q_{il}^{23} & q_{il}^{24} \\ & & q_{jj}^{11} & q_{jj}^{12} & q_{jk}^{13} & q_{jk}^{14} & q_{jl}^{13} & q_{jl}^{14} \\ & & & q_{jj}^{22} & q_{jk}^{23} & q_{jk}^{24} & q_{jl}^{23} & q_{jl}^{24} \\ & & & & q_{kk}^{33} & q_{kk}^{34} & q_{kl}^{33} & q_{kl}^{34} \\ & sym & & & & q_{kk}^{44} & q_{kl}^{34} & q_{kl}^{44} \\ & & & & & & q_{ll}^{33} & q_{ll}^{34} \\ & & & & & & & q_{ll}^{44} \end{bmatrix}. \tag{10}$$

By evaluation of Q_z based on the equations in the Table. 3, and inserting Q_z into Eq. (10), the 2nd order statistics of double difference measurements are derived as:

$$\mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} = 2\sigma^2 \Delta t_{12}^p (1 - e^{-\frac{h_{ij}}{L}}), \tag{11}$$

and

$$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} = \frac{1}{2}\sigma^2 \left(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\right)\left(e^{-\frac{h_{ik}}{L}} + e^{-\frac{h_{jl}}{L}} - e^{-\frac{h_{il}}{L}} - e^{-\frac{h_{jk}}{L}}\right). \tag{12}$$

Table 4: Overview of these 2nd order statistics of double differences
for **spatially correlated** fractional Brownian processes.

Description	Symbol	Statistics
Generic case	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(e^{-\frac{h_{ik}}{L}} + e^{-\frac{h_{jl}}{L}} - e^{-\frac{h_{il}}{L}} - e^{-\frac{h_{jk}}{L}}\Big)$
same arc, same epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_1t_2}\} = \mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\}$	$2\sigma^2\Delta t_{12}^p(1 - e^{-\frac{h_{ij}}{L}})$
same arc, different epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_3t_4}\}$	$\sigma^2\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)(1 - e^{-\frac{h_{ij}}{L}})$
same arc, different epochs same reference epoch (t_0)	$\mathrm{C}\{\underline{z}_{ij}^{t_0t_1}, \underline{z}_{ij}^{t_0t_2}\}$	$\sigma^2\Big(\Delta t_{01}^p + \Delta t_{02}^p - \Delta t_{12}^p\Big)(1 - e^{-\frac{h_{ij}}{L}})$
same reference point (r)	$\mathrm{C}\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_3t_4}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(1 + e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{ri}}{L}} - e^{-\frac{h_{rj}}{L}}\Big)$
same reference point (r), same reference epoch (t_0)	$\mathrm{C}\{\underline{z}_{ri}^{t_0t_1}, \underline{z}_{rj}^{t_0t_2}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{02}^p + \Delta t_{01}^p - \Delta t_{12}^p\Big)\Big(1 + e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{ri}}{L}} - e^{-\frac{h_{rj}}{L}}\Big)$
same reference point (r) same epochs	$\mathrm{C}\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p\Big(1 + e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{ri}}{L}} - e^{-\frac{h_{rj}}{L}}\Big)$
same epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p\Big(e^{-\frac{h_{ik}}{L}} + e^{-\frac{h_{jl}}{L}} - e^{-\frac{h_{il}}{L}} - e^{-\frac{h_{jk}}{L}}\Big)$

Note that in case of a spatially uncorrelated signal (i.e., $L \to \epsilon$), the exponential component equals zero and the the Eq. (11) reduces to:

$$\mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} = 2\sigma^2\Delta t_{12}^p, \tag{13}$$

and the covariance components of Eq. (12) will be zero in a general case. Note that, if $h \neq 0$ the exponential components $e^{-h/\epsilon} = 0$, but when $h = 0$, then we get $e^{-h/\epsilon} = 1$. Therefore, for spatially uncorrelated signals, the covariance between DD combinations are zero only when the two DD measurements share no common benchmark. In other cases, there is a correlation induced by the common benchmark. For example, if the two DDs share the same reference point, then $i = k$ and the covariance of Eq. (12) for spatially uncorrelated components (i.e., when $L = \epsilon$) is

$$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{il}^{t_3t_4}\} = \frac{1}{2}\sigma^2\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(1 + 0 - 0 - 0\Big) = \frac{1}{2}\sigma^2\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big). \tag{14}$$

Especial cases of Eq. (12) for different choices of i, j, k, l, t_1, t_2, t_3, and t_4 have been given in the tables 4 and 5 for spatially correlated and spatially uncorrelated signals, receptively.

A.3 Variogram model for double differences

Variogram between two double differences $\underline{z}_{ij}^{t_1t_2}$ and $\underline{z}_{kl}^{t_3t_4}$ is defined as (Wackernagel, 1995; Chilès and Delfiner, 1999)

$$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} = \frac{1}{2}\mathrm{E}\{\big(\underline{z}_{ij}^{t_1t_2} - \underline{z}_{kl}^{t_3t_4}\big)^2\} = \frac{1}{2}\mathrm{D}\{\underline{z}_{ij}^{t_1t_2} - \underline{z}_{kl}^{t_3t_4}\}. \tag{15}$$

Table 5: Overview of these 2nd order statistics of double differences
for **spatially uncorrelated** fractional Brownian processes.

Description	Symbol	Statistics
Generic case	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}$	0
same arc, same epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_1t_2}\} = \mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\}$	$2\sigma^2\Delta t_{12}^p$
same arc, different epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_3t_4}\}$	$\sigma^2\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)$
same arc, different epochs same reference epoch (t_0)	$\mathrm{C}\{\underline{z}_{ij}^{t_0t_1}, \underline{z}_{ij}^{t_0t_2}\}$	$\sigma^2\Big(\Delta t_{01}^p + \Delta t_{02}^p - \Delta t_{12}^p\Big)$
same reference point (r)	$\mathrm{C}\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_3t_4}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)$
same reference point (r), same reference epoch (t_0)	$\mathrm{C}\{\underline{z}_{ri}^{t_0t_1}, \underline{z}_{rj}^{t_0t_2}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{02}^p + \Delta t_{01}^p - \Delta t_{12}^p\Big)$
same reference point (r) same epochs	$\mathrm{C}\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p$
same epochs	$\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_1t_2}\}$	0

To derive an analytical model for variograms between double differences, the dispersion $\mathrm{D}\{\underline{z}_{ij}^{t_1t_2} - \underline{z}_{kl}^{t_3t_4}\}$ should be evaluates. Based on the linear error propagation law we have:

$$\mathrm{D}\{\underline{z}_{ij}^{t_1t_2} - \underline{z}_{kl}^{t_3t_4}\} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} & \mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} \\ \mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} & \mathrm{D}\{\underline{z}_{kl}^{t_3t_4}\} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{16}$$

Inserting the Eq. (16) into Eq. (15) gives the variogram function as:

$$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} = \frac{1}{2}\Big(\mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\} + \mathrm{D}\{\underline{z}_{kl}^{t_3t_4}\} - 2\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}\Big) \tag{17}$$

The analytical expression of $\mathrm{D}\{\underline{z}_{ij}^{t_1t_2}\}$, $\mathrm{D}\{\underline{z}_{kl}^{t_3t_4}\}$, and $\mathrm{C}\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}$ is given by Eqs. 11 and (12). By inserting them into Eq. (17), we derive the generic analytical expression for the variogram as

$$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\} = \sigma^2\Delta t_{12}^p(1 - e^{-\frac{h_{ij}}{L}}) + \sigma^2\Delta t_{34}^p(1 - e^{-\frac{h_{kl}}{L}}) - ... \tag{18}$$

$$...\frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(e^{-\frac{h_{ik}}{L}} + e^{-\frac{h_{jl}}{L}} - e^{-\frac{h_{il}}{L}} - e^{-\frac{h_{jk}}{L}}\Big).$$

Especial cases of Eq.(18) for different choices of i, j, k, l, t_1, t_2, t_3, and t_4 have been given in the tables 6 and Table 7 for spatially correlated and uncorrelated signals, receptively.

Table 6: Analytical variogram model for double differences
for **spatially correlated** fractional Brownian processes.

Description	Symbol	Variogram model
Generic case	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}$	$\sigma^2\Delta t_{12}^p(1-e^{-\frac{h_{ij}}{L}}) + \sigma^2\Delta t_{34}^p(1-e^{-\frac{h_{kl}}{L}})...$ $... - \frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(e^{-\frac{h_{ik}}{L}} + e^{-\frac{h_{jl}}{L}} - e^{-\frac{h_{il}}{L}} - e^{-\frac{h_{jk}}{L}}\Big)$
same arc, same epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_1t_2}\}$	0
same arc, different epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_3t_4}\}$	$\sigma^2\Big(\Delta t_{12}^p + \Delta t_{34}^p - \Delta t_{14}^p - \Delta t_{23}^p + \Delta t_{13}^p + \Delta t_{24}^p\Big)(1-e^{-\frac{h_{ij}}{L}})$
same arc, different epochs same reference epoch (t_0)	$\gamma\{\underline{z}_{ij}^{t_0t_1}, \underline{z}_{ij}^{t_0t_2}\}$	$\sigma^2\Delta t_{12}^p(1-e^{-\frac{h_{ij}}{L}})$
same reference point (r)	$\gamma\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_3t_4}\}$	$\sigma^2\Delta t_{12}^p(1-e^{-\frac{h_{ri}}{L}}) + \sigma^2\Delta t_{34}^p(1-e^{-\frac{h_{rj}}{L}})...$ $- \frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)\Big(1 + e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{ri}}{L}} - e^{-\frac{h_{rj}}{L}}\Big)$
same reference point (r), same reference epoch (t_0)	$\gamma\{\underline{z}_{ri}^{t_0t_1}, \underline{z}_{rj}^{t_0t_2}\}$	$\sigma^2\Delta t_{01}^p(1-e^{-\frac{h_{ri}}{L}}) + \sigma^2\Delta t_{02}^p(1-e^{-\frac{h_{rj}}{L}})...$ $- \frac{\sigma^2}{2}\Big(\Delta t_{02}^p + \Delta t_{01}^p - \Delta t_{12}^p\Big)\Big(1 + e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{ri}}{L}} - e^{-\frac{h_{rj}}{L}}\Big)$
same reference point (r) same epochs	$\gamma\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p(1-e^{-\frac{h_{ij}}{L}})$
same epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p\Big(2 - e^{-\frac{h_{ij}}{L}} - e^{-\frac{h_{kl}}{L}} - e^{-\frac{h_{ik}}{L}} - e^{-\frac{h_{jl}}{L}} + e^{-\frac{h_{il}}{L}} + e^{-\frac{h_{jk}}{L}}\Big)$

Table 7: Analytical variogram model for double differences
for **spatially uncorrelated** fractional Brownian processes.

Description	Symbol	Variogram model
Generic case	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_3t_4}\}$	$\sigma^2\Delta t_{12}^p + \sigma^2\Delta t_{34}^p$
same arc, same epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_1t_2}\}$	0
same arc, different epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{ij}^{t_3t_4}\}$	$\sigma^2\Big(\Delta t_{12}^p + \Delta t_{34}^p - \Delta t_{14}^p - \Delta t_{23}^p + \Delta t_{13}^p + \Delta t_{24}^p\Big)$
same arc, different epochs same reference epoch (t_0)	$\gamma\{\underline{z}_{ij}^{t_0t_1}, \underline{z}_{ij}^{t_0t_2}\}$	$\sigma^2\Delta t_{12}^p$
same reference point (r)	$\gamma\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_3t_4}\}$	$\sigma^2\Delta t_{12}^p + \sigma^2\Delta t_{34}^p - \frac{\sigma^2}{2}\Big(\Delta t_{14}^p + \Delta t_{23}^p - \Delta t_{13}^p - \Delta t_{24}^p\Big)$
same reference point (r), same reference epoch (t_0)	$\gamma\{\underline{z}_{ri}^{t_0t_1}, \underline{z}_{rj}^{t_0t_2}\}$	$\frac{\sigma^2}{2}\Big(\Delta t_{02}^p + \Delta t_{01}^p - \Delta t_{12}^p\Big)$
same reference point (r) same epochs	$\gamma\{\underline{z}_{ri}^{t_1t_2}, \underline{z}_{rj}^{t_1t_2}\}$	$\sigma^2\Delta t_{12}^p$
same epochs	$\gamma\{\underline{z}_{ij}^{t_1t_2}, \underline{z}_{kl}^{t_1t_2}\}$	$2\sigma^2\Delta t_{12}^p$

B

References

Chilès, J.-P. and P. Delfiner (1999). *Geostatistics: modeling spatial uncertainty*. Wiley Series in Probability and Statistics. New York: John Wiley & Sons.

Coleman, T. and Y. Li (1996). An interior, trust region approach for nonlinear minimization subject to bounds. *SIAM Journal on Optimization 6*, 418445.

Samiei-Esfahany, S. and H. Bähr (2015). Research and development project for geodetic deformation monitoring, for long-term study on anomalous time-dependent subsidence in the Wadden sea region. Technical report, Nederlanse Aardolie Maatschappij B.V., Assen, Netherlands. EP201505216980.

van Leijen, F., S. Samiei-Esfahany, H. van der Marel, and R. Hanssen (2017). Uniformization of geodetic data for deformation analysis; contribution to the research project: Second phase of the long-term subsidence study in the Wadden Sea region (LTS2). Technical report, Delft University of Technology, Delft, Netherlands.

Wackernagel, H. (1995). *Multivariate Geostatistics: An Introduction with Applications*. Berlin: Springer-Verlag.

Yaglom, A. M. (1962). *An Introduction to the Theory of Stationary Random Functions* (1st ed.). Prentice-Hall.