EP2015-06209300
Derivation of a Scale Dependent Pressure Diffusion Equation

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1 Hydraulic Conductivity - Power Laws and Scale Dependence

The hydraulic conductivity of a porous permeable rock depends on the random distribution of pore spaces and pore throats and how these connect over various length scales to provide fluid flow paths of greater or lesser resistance. Generalized central limit theory states that accumulated random processes will converge to a stable distrbution [Mandelbrot, 1960; Nolan, 2015]. This is sometimes mistakenly assumed to mean that all random processes will converge to a normal/gaussian probability distribution, however, there are other stable distributions. With the exception of the normal/gaussian distribution these other stable distributions have power law or 'heavy' tails, that is they are not exponentially bounded. That is not to say that they are defined as pure power laws, but that they approach such for a significant part of their domain and can therefore be usefully approximated as such, especially close to their bounds.

Power law distributions have limited statistical moments, the power law exponent has to exceed the moment order plus one, for the moment to be finite, i.e for power law exponents less than 3 the distribution has no finite variance, for exponents less than 2 the distribution has no finite mean (or higher moments). Another signature property of power law distributions is their scale invariance and hence they are associated with self similar phenomena. The somewhat counter-intuitive outcome of such scale invariance is that it yields scale dependent parameters, e.g. the amplitude of a self-similar rough surface, with a random power law distribution, will depend on the domain scale being considered. This means that for properties that are power law distributed there is no representative scale, for three-dimensional properties of this type, there is no representative elementary volume [REV]. This can have a profound impact on their analysis, as standard methods of analytical and numerical calculation implicitly assume a representative subscale (it is the basis on which calculus is built and is explicit in finite difference and finite element computation).

Natural hydraulic conductivity/permeability in rock formations is generally observed to be 'heavy tailed', i.e. far from normally distributed, even in some of the most homogeneous seeming materials [Neuman et al., 2000; Sposito, 2008]. It is also observed that hydraulic conductivity appears to be scale dependent, larger samples of the same formation seem to have higher conductivities than smaller samples [Neuman, 1990; Schulze-Makuch & Cherkauer, 1998; Neuman et al., 2000; Sposito, 2008; Fallico, 2014]. Further evidence points to correlation length scales for rock hydraulic conductivity that appear to be unbounded, or at least too large to be considered as significantly smaller than the observation scale, and hence precluding any consideration of an REV, [Federico & Tartakovsky, 2000]. The hydraulic conductivity correlation length scale has a direct connection to the bulk conductivity as fluid flow at any location is not just governed by local conductivity, but by its connectivity as well (no fluid can flow through a domain, however conductive it may be, if it is surrounded and sealed by a non-conductive perimeter).

There is a significant literature on chemical contaminant diffusivity in materials where the permeability is power law distributed, but this is posited on the power law being applied to stopping times, essentially governing the random walk flight of the contaminant molecules [Benson et al., 2000a; Benson et al., 2000b; Benson et al., 2001; Schumer et al., 2001; Berkowitz et al., 2006, Wheatcraft & Meerschaert, 2008]. This paper though attempts to investigate anomalous fluid pressure diffusion based on power law distributed hydraulic conductivity. It does so by first deriving a Darcy type flux gradient law for such a conductivity field and then combing this with a standard flux divergence relationship based on conservation of mass.

2 Scale Dependent Darcy's law

Darcy's law in its simple form relates fluid flow through a length of permeable rock, of constant cross-section, to the pressure drop, scaled by a hydraulic/resistance factor

$$p(x_0) - p(x_1) = -RQ \tag{1}$$

where p is pressure, x_0 and x_1 are positions along direction $\hat{\mathbf{x}}$, Q is a volume flow rate and R is a hydraulic resistance term

$$R = \frac{\|x_1 - x_0\|\eta}{kA} \tag{2}$$

with η being the fluid dynamic viscosity, k the rock hydrulic permeability and A the cross-sectional area. It's a flux gradient law, of the same class as Ohm's law or Fourier's law.

In three dimensions Darcy's law is generalized to the following forms

$$\mathbf{q} = \mathbf{K} \cdot \nabla p \tag{3}$$

$$\nabla p = \mathbf{W} \cdot \mathbf{q} \tag{4}$$

where $\mathbf{q}$ is the fluid flux vector, $\mathbf{K}$ is the hydraulic conductivity tensor and $\mathbf{W}$ is its inverse, the hydraulic resistivity tensor, $\mathbf{W} = \mathbf{K}^{-1}$. In this analysis assusme that hydraulic conductivity/resistivity is isotropic

$$\mathbf{K} = K\mathbf{I} \quad ; \quad \mathbf{W} = W\mathbf{I} \tag{5}$$

Consider a simple, one dimensional flow type flow

$$p(x_0) - p(x_1) = \int_{x_0}^{x_1} W \, dx \; q_x \tag{6}$$

The pore fluid pressure is a location defined quantity, it is spatially varying but not scale dependent, this allows it to be defined for an arbitrarily small REV (applying local homogenisation) and has a well defined gradient at a point. This allows for the application of a Taylor series expansion to the pressure term

$$p(x_0) - \sum_{n=0}^{\infty} \frac{\partial^n p(x_0)}{\partial x^n} \frac{(x_1 - x_0)^n}{n!} = \int_{x_0}^{x_1} W \, dx \, q_x \tag{7}$$

The Taylor series converges rapidly for smooth functions and if the function is linear is exact after two terms (the constant value and first derivative term). For a smooth function linearity can always be approached by simply shrinking the representative elementary volume [REV] considered.

The hydraulic resistance (i.e. the integrated resistivity) for a permeable material though, is always scale dependent - the more material the fluid has to pass through, the greater the hydraulic resistance. If the hydraulic resistivity is constant then determining the hydraulic resistivity is trivial

$$R_x = \int_{\Delta x} W \, dx = W \, \Delta x \tag{8}$$

where $\Delta x = x_1 - x_0$. If the hydraulic resistivity is spatially variable but independent of scale, then, just as in the case of the pressure function, a Taylor series expansion can again be applied

$$R_x = \int_{\Delta x} W \, dx = \sum_{n=0}^{\infty} \frac{\partial^n R_x(x_0)}{\partial x^n} \frac{(x_1 - x_0)^n}{n!} \tag{9}$$

note that

$$R_x(x_0) = 0 \;\; ; \;\; \frac{\partial R_x(x_0)}{\partial x} = W(x_0) \tag{10}$$

However, as has been discussed above the hydraulic conductivity/resistivity is observed to be scale dependent. Typically the hydraulic conductivity, size scale relationship is cited as

$$K(\Delta x) = K_0 + C(\Delta x)^{\gamma} \tag{11}$$

where K_0 is a constant background hydraulic conductivity; C is a constant of appropriate dimensionality; and γ is a small value between 0 and 1 [Schulze-Makuck & Cherkauer, 1998; Fallico, 2014]. [N.B. The K_0 is often neglected as investigators are generally determining the relationship in a log-log space]. In these cases the REV/correlation length scales are not small compared to the problem scale and the standard Taylor series expansion will no longer converge rapidly and can't appropriately be applied. In recent years though, other more general Taylor series type expansions have been developed based on non-integer derivative operators - so called 'fractional derivatives' [Odibat & Shawagfeh, 2007; Truilljo et al., 1999]. Fractional derivative type equations appear to be a more natural framework for the analysis and representation of scale dependent phemomena [Metzler & Klafter, 2004; Wheatcraft & Meerschaert, 2008], and the following derivation is based on some of the ideas developed in this latter paper.

The generalized Taylor expansion derived by Odibat & Shawagfeh [Odibat & Shawagfeh, 2007] is defined as

$$F(x + \Delta x) = \sum_{n=0}^{\infty} D_x^{\alpha\,(n)} F(x+) \frac{(\Delta x)^n}{\Gamma(n\alpha + 1)} \quad ; \quad 0 < \alpha \leq 1 \tag{12}$$

Here D_x^α is the Caputo fractional derivative of order α at x; $D_x^{\alpha\,(n)}$ denotes the n-fold action of this fractional derivative operator; $x+$ signifies that where limits need to be taken they approach from the positive side; and Γ is the Gamma function. The Caputo fractional derivative is defined in integral form as

$$D_x^\alpha F(x + \Delta x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{x+\Delta x} F(x + \Delta x - u)\,(u - x)^{-\alpha}\,du \quad ; \quad 0 < \alpha \leq 1 \tag{13}$$

This generalized Taylor expansion converges rapidly for functions that scale with order close to α and for functions of the type

$$f(x + \Delta x) \propto f(x) + (x + \Delta x)^\alpha \tag{14}$$

the series converges completely in the first two terms.

Returning then to Darcy's law we can consider a permeable material where hydraulic resistance scales with $(\Delta x)^{1-\gamma}$ (as would be expected for a material where hydraulic conductivity scales as $(\Delta x)^{\gamma}$. This seems intuitively reasonable, for a material with constant hydraulic conductivity the hydraulic resistance would grow proportionately with the length scale of the flow path, $R \propto \Delta x$. But for a material where conductivity was observed to increase with scale, then the hydraulic resistance would be expected to grow more slowly.

$$R_x = \int_{\Delta x} W \, dx \propto (\Delta x)^{1-\gamma} \tag{15}$$

Expressing this in the form of a generalized Taylor expansion gives

$$R(\Delta x) = \sum_{n=0}^{\infty} D_x^{\alpha \ (n)} R(x+) \frac{(\Delta x)^n}{\Gamma(n\alpha + 1)} \quad ; \quad 0 < \alpha \leq 1 \tag{16}$$

Setting the order of the fractional derivative operator $\alpha = 1 - \gamma$ will allow the series to converge exactly with just the first two terms. The Darcy type flux gradient law for such a material could then be expressed as

$$\frac{\partial p(x)}{\partial x} = D_x^{1-\gamma} R(x+) \frac{q_x}{(\Delta x)^{\gamma} \Gamma(2-\gamma)} \tag{17}$$

In this form it becomes apparent that the flux-gradient relationship is now dependent on the scale under consideration. As the material is assumed to be isotropic and therefore isotropic in its scale dependence

$$D_x^{1-\gamma} R(x+) = D_y^{1-\gamma} R(y+) = D_z^{1-\gamma} R(z+) \tag{18}$$

the more general, three-dimensional Darcy type law can be written

$$\nabla p = D_x^{1-\gamma} R(x+) \frac{\mathbf{q}}{(\Delta x)^{\gamma} \Gamma(2-\gamma)} \tag{19}$$

The material hydraulic conductivity can be considered in a manner analogous to that of the hydraulic resistance. That is the hydraulic conductivity (note, not the conductance) can similarly be written as a generalized Taylor expansion

$$K(\Delta x) = \sum_{n=0}^{\infty} D_x^{\alpha\ (n)} K(x+) \frac{(\Delta x)^n}{\Gamma(n\alpha + 1)} \quad ; \quad 0 < \alpha \leq 1 \tag{20}$$

As $K \propto \gamma$, rapid convergence in this expansion can be achieved by setting $\alpha = \gamma$, and here, as before, complete convergence would only require the first two terms in the series. And again, assuming isotropy the Darcy type flow law for a material with scale dependent hydraulic conductivity, $K(\Delta x) \propto (\Delta x)^\gamma$, where $0 < \gamma \leq 1$

$$\mathbf{q} = \left(K_0 + D_x^\gamma K(x+) \frac{(\Delta x)^\gamma}{\Gamma(\gamma + 1)} \right) \nabla p \tag{21}$$

3 Flux Divergence

The mass conservation law using an Eulerian description is

$$\rho \nabla \cdot \mathbf{q} = -\frac{\partial M}{\partial t} \tag{22}$$

where ρ is fluid density and M is the mass of fluid per unit volume of media. The rate of change in M can be written as

$$\frac{\partial M}{\partial t} = \rho \frac{\partial \phi}{\partial t} + \phi \frac{\partial \rho}{\partial t} \tag{23}$$

If it can be assumed that changes in porosity, ϕ, and density, ρ, are dominated by changes in pressure, and that the pore space and fluid behave linear elastically then:-

$$\frac{\partial M}{\partial t} = \rho \phi (\beta_f + \beta_\phi) \frac{\partial \rho}{\partial t} \tag{24}$$

where β_f and β_ϕ are respectively the fluid and pore volume compressibilities. These can be defined as follows

$$\beta_f = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \;\; ; \;\; \beta_\phi = \frac{1}{\phi}\frac{\partial \phi}{\partial p} \tag{25}$$

Combining these equations yields the following flux divergence relationship

$$\nabla \cdot \mathbf{q} = -\phi(\beta_f + \beta_\phi)\frac{\partial p}{\partial t} \tag{26}$$

4 A Scale Dependent Pressure Diffusion Equation

Inserting the scale dependent Darcy type flux gradient law, expressed in terms of conductivity, into the flux divergence relationship gives

$$\nabla \cdot \left[\left(K_0 + D_x^\gamma K(x+) \frac{(\Delta x)^\gamma}{\Gamma(\gamma+1)} \right) \nabla p \right] = -\phi(\beta_f + \beta_\phi)\frac{\partial p}{\partial t} \tag{27}$$

This can be rewritten as

$$\left(K_0 + D_x^\gamma K(x+) \frac{(\Delta x)^\gamma}{\Gamma(\gamma+1)} \right) \nabla^2 p + \left(D_x^{\gamma+1} K(x+) \frac{(\Delta x)^\gamma}{\Gamma(\gamma+1)} \right) \nabla p = -\phi(\beta_f + \beta_\phi)\frac{\partial p}{\partial t} \tag{28}$$

(by recognising that the scale, Δx, does not depend on the location, x).

At first sight, this may appear relatively difficult to detemine, however, if the order of the fractional derivative operator is matched properly to the scale dependency of the conductivity, i.e. $K(\Delta x) \propto (\Delta x)^\gamma$, then the term $D_x^\gamma K(x+)$ should be constant and henece $D_x^{\gamma+1} K(x+)$ will be zero. This then yields

$$\left(K_0 + D_x^\gamma K(x+)\frac{(\Delta x)^\gamma}{\Gamma(\gamma+1)}\right)\nabla^2 p = -\phi(\beta_f + \beta_\phi)\frac{\partial p}{\partial t} \tag{29}$$

which is essentially just an ordinary linear diffusion equation but with a scale dependent diffusivity term. Assuming that the scale dependent conductivity is essentially a power law realtionship, and that the order of the fracional derivative has been matched, the scale dependent diffusivity term will reduce to a simple coefficient. This will result in a standard diffusion equation with a single diffusivity term that will simply depend on the scale being considered.

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